Functional Analysis

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Lecture 2

Spaces of continuous functions

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Spaces of continuous functions

Let Ω be a fixed topological space.

Continuous functions $C(\Omega) := \{x : \Omega \to \mathbb{F} \text{ is continuous}\}$ with values in the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and

$$(x+y)(t) := x(t)+y(t), \qquad (\lambda x)(t) := \lambda x(t) \qquad \left(\begin{array}{c}\text{pointwise}\\ \text{operations}\end{array}\right)$$

where $x, y \in C(\Omega)$, $\lambda \in \mathbb{F}$, is a linear space over $\mathbb{F}$.

Continuous and bounded functions:

$$\begin{aligned} C_b(\Omega) &:= \{x \in C(\Omega) : \exists_M \forall_{t\in\Omega} |x(t)| < M\} \\ &= \{x \in C(\Omega) : \sup_{t\in\Omega} |x(t)| < \infty\} \end{aligned}$$

form a linear subspace of $C(\Omega)$, which is equipped with the so-called **supremum norm**

$$\|x\|_\infty := \sup_{t\in\Omega} |x(t)|.$$

Fact. Convergence in $\|\cdot\|_\infty \equiv$ **uniform convergence**

$$
\begin{aligned}
x_n \xrightarrow{\|\cdot\|_\infty} x \iff & \lim_{n\to\infty} \|x_n - x\|_\infty = 0 \\
\iff & \lim_{n\to\infty} \sup_{t\in\Omega} |x_n(t) - x(t)| = 0 \\
\iff & \forall_{\varepsilon>0}\, \exists_{N\in\mathbb{N}}\, \forall_{n>N}\ \sup_{t\in\Omega} |x_n(t) - x(t)| < \varepsilon \\
\iff & \forall_{\varepsilon>0}\, \exists_{N\in\mathbb{N}}\, \forall_{n>N}\ \forall_{t\in\Omega}\ |x_n(t) - x(t)| < \varepsilon \\
\overset{def}{\iff} & x_n \rightrightarrows x
\end{aligned}
$$

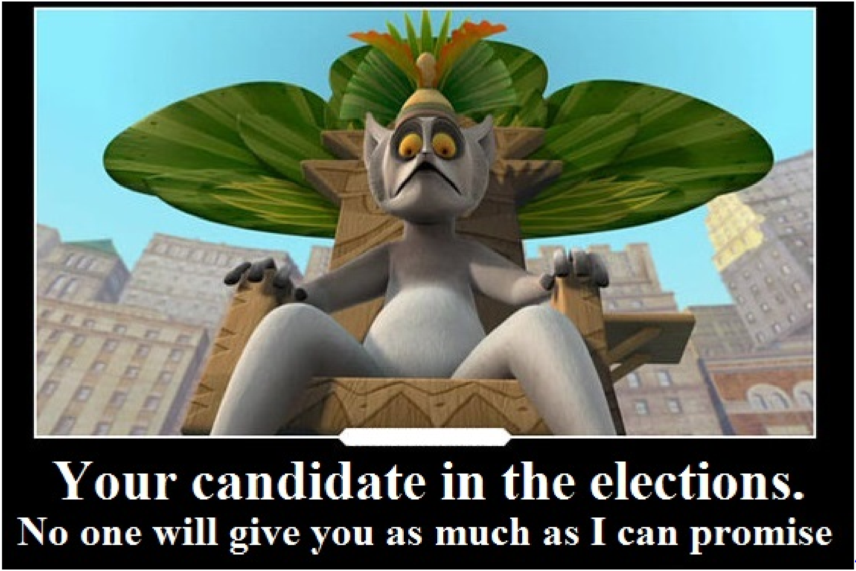
where symbol $\rightrightarrows$ denotes the uniform convergence.

Ex. The sequence $x_n(t) = t^n$ of functions on $[0,1]$ is pointwise convergent to $x(t) = \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases}$. But $\{x_n\}_{n=1}^{\infty} \subseteq C(\Omega)$ is not convergent in the norm $\|\cdot\|_\infty$. In particular,

> ***uniformly convergent sequence of continuous functions has to be convergent to a continuous function!***

Prop. $C_b(\Omega)$ with the norm $\|x\|_\infty = \sup_{t\in\Omega} |x(t)|$ is a Banach space.

Proof: Let $\{x_n\}_{n=1}^{\infty} \subseteq C_b(\Omega)$ be Cauchy. For each $t \in \Omega$



Prop. $C_b(\Omega)$ with the norm $\|x\|_\infty = \sup\limits_{t\in\Omega}|x(t)|$ is a Banach space.

Proof: Let $\{x_n\}_{n=1}^\infty \subseteq C_b(\Omega)$ be Cauchy. For each $t \in \Omega$

$$|x_n(t) - x_m(t)| \leqslant \sup_{s\in\Omega}|x_n(s) - x_m(s)| = \|x_n - x_m\|_\infty \overset{n,m\to\infty}{\longrightarrow} 0.$$

Hence the sequence $\{x_n(t)\}_{n=1}^\infty$ of scalars is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete, there is $x(t) \in \mathbb{F}$ such, that $x_n(t) \to x(t)$ w $\mathbb{F}$. Thus we obtain a function $\Omega \ni t \mapsto x(t) \in \mathbb{F}$, which is our "candidate for the limit" of the sequence $\{x_n\}_{n=1}^\infty$. For each $t \in \Omega$ we have

$$|x_n(t) - x(t)| = \lim_{m\to\infty}|x_n(t) - x_m(t)| \leqslant \lim_{m\to\infty}\|x_n - x_m\|_\infty.$$

Hence

$\{x_n\}_{n=1}^\infty$ jest Cauchy

$$\lim_{n\to\infty}\|x_n - x\|_\infty = \lim_{n\to\infty}\sup_{t\in\Omega}|x_n(t) - x(t)| \leqslant \lim_{n\to\infty}\lim_{m\to\infty}\|x_n - x_m\|_\infty = 0.$$

Whence $x_n \overset{\|\cdot\|_\infty}{\longrightarrow} x$. The uniform limit of continuous functions is continuous. Therefore $x \in C(\Omega)$. Moreover, x is bounded because

$$\|x\|_\infty \leqslant \|x - x_n\|_\infty + \|x_n\|_\infty < \infty. \qquad \text{Thus } x \in C_b(\Omega).$$ ■

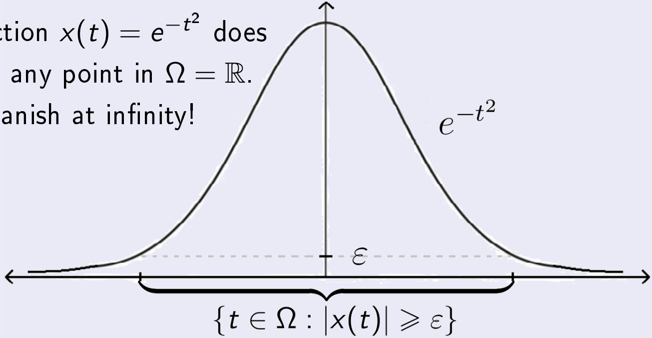
A continuous function on a compact set reaches its limits!

Cor. If Ω compact, then $C(\Omega) = C_b(\Omega)$ and $\|x\|_\infty = \max_{t\in\Omega} |x(t)|$ (the function $|x(t)|$ attains its maximum and in particular is bounded)

Continuous functions that vanish at infinity

$$C_0(\Omega) := \Big\{ x \in C(\Omega) : \forall_{\varepsilon>0}\ \{t \in \Omega : |x(t)| \geqslant \varepsilon\} \text{ is compact} \Big\}$$

Ex. The function $x(t) = e^{-t^2}$ does not vanish at any point in $\Omega = \mathbb{R}$. But it does vanish at infinity!



Rem. If Ω is compact, then $C_0(\Omega) = C(\Omega) = C_b(\Omega)$!

Prop. $C_0(\Omega)$ is a closed subspace of the Banach space $(C_b(\Omega), \|\cdot\|_\infty)$. Hence $(C_0(\Omega), \|\cdot\|_\infty)$ is a Banach space.

Proof: Let $x, y \in C_0(\Omega)$ and $\varepsilon > 0$. Note that

$$\underbrace{\{t : |x(t) + y(t)| \geqslant \varepsilon\}}_{\text{closed}} \subseteq \underbrace{\{t : |x(t)| \geqslant \varepsilon/2\} \cup \{t : |y(t)| \geqslant \varepsilon/2\}}_{\text{compact as a union of two compacts}}.$$

Hence $\{t : |x(t) + y(t)| \geqslant \varepsilon\}$ is compact, as a closed susbet of a compact set. Hence $x + y \in C_0(\Omega)$. For $\lambda \in \mathbb{F}$ the set $\{t : |\lambda x(t)| \geqslant \varepsilon\} = \{t : |x(t)| \geqslant \frac{\varepsilon}{|\lambda|}\}$ is compact, so $\lambda x \in C_0(\Omega)$. Since $|x(t)|$ is continous on the compact set $\{t : |x(t)| \geqslant \varepsilon\}$, we get $\|x\|_\infty = \max_{t\in\Omega} |x(t)| < \infty$. Thus $C_0(\Omega) \subseteq C_b(\Omega)$ is a subspace.

„Closedness”: Let $\{x_n\}_{n=1}^\infty \subseteq C_0(\Omega)$ be convergent to some $x \in C_b(\Omega)$. For large $n \in \mathbb{N}$ we have $\|x_n - x\|_\infty < \frac{\varepsilon}{2}$ and then

$$\{t : |x(t)| \geqslant \varepsilon\} \subseteq \{t : |x_n(t)| \geqslant \varepsilon/2\}.$$

Thus $\{t : |x(t)| \geqslant \varepsilon\}$ is compact, as a closed subset of a compact set. Hence $x \in C_0(\Omega)$. Accordingly, $C_0(\Omega)$ is closed in $C_b(\Omega)$. ■

On a discrete spaces all functions are continuous!
Sequences are functions on the set $\mathbb{N}$!

Cor. If $\Omega = \mathbb{N}$ is equipped with discrete topology, then we identify $C_b(\Omega)$ with the **space of bounded sequences**:

$$\ell^\infty := \{x = (x(1), x(2), \dots) : \sup_{k\in\mathbb{N}} |x(k)| < \infty\}$$

equipped with the norm $\|x\|_\infty := \sup_{k\in\mathbb{N}} |x(k)|$. Similarly, $C_0(\Omega)$ can be identified with the **space of sequences convergent to zero**:

$$c_0 := \{x = (x(1), x(2), \dots) : \lim_{k\to\infty} x(k) = 0\}.$$

Proof: A function $x : \mathbb{N} \to \mathbb{F}$ vanishes at infinity if and only if the sequence $\{x(k)\}_{k=1}^\infty$ converges to zero:

in discrete spaces

$$x \in C_0(\mathbb{N}) \iff \forall_{\varepsilon>0}\ \{k \in \mathbb{N} : |x(k)| \geqslant \varepsilon\} \quad \text{compact} = \text{finite}$$

$$\iff \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{k>N} |x(k)| < \varepsilon \iff x \in c_0.$$

We also consider the **space of convergent sequences**:

$$c := \{x = (x(1), x(2), \dots) : \exists_{x(\infty)\in\mathbb{F}} \lim_{k\to\infty} x(k) = x(\infty)\}.$$

As convergent sequences are bounded and limit preserves linear combinations, c is a linear subspace of ℓ^∞. Also c is closed in ℓ^∞ (to prove it, you have to show that "*a convergent sequence of convergent sequences converges to a convergent sequence*").

Cor. We have the following Banach spaces

$$c_0 \subseteq c \subseteq \ell^\infty$$

with the norm $\|x\|_\infty := \sup_{k\in\mathbb{N}} |x(k)|$.

The **support of a function** $x : \Omega \to \mathbb{F}$ is the closed set

$$\operatorname{supp}(x) := \overline{\{t \in \Omega : x(t) \neq 0\}}.$$

Ex. If $\Omega = (0, +\infty)$ and $x(t) = \sin(1/t)$, then $\operatorname{supp}(x) = \Omega$.

Lem. **Compactly supported continuous functions**

$$C_c(\Omega) := \{x \in C(\Omega) : \operatorname{supp}(x) \text{ is compact}\}$$

form a linear subspace of $C_0(\Omega)$.

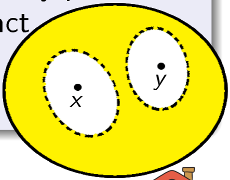
Proof: $C_c(\Omega) \subseteq C_0(\Omega)$, because $\{t \in \Omega : |x(t)| \geqslant \varepsilon\} \subseteq \operatorname{supp}(x)$ and a closed subset of a compact set is compact. Moreover, $\operatorname{supp}(\lambda x) = \operatorname{supp}(x)$ for $\lambda \neq 0$ and

$$\operatorname{supp}(x + y) \subseteq \operatorname{supp}(x) \cup \operatorname{supp}(y),$$

when $C_c(\Omega)$ is a linear space. ∎

Def. A topological space Ω is **locally compact** if any point in Ω has an open neighborhood, whose closure is compact.

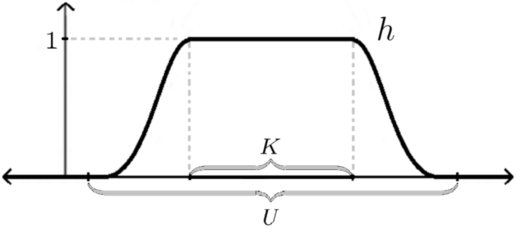
The space Ω is **Hausdorff** if any two distinct points in Ω have disjoint open neighborhoods.



Ex. Every open or closed subset of $\mathbb{R}^n$ is a locally compact Hausdorff space. Every metric space is Hausdorff.

Theorem (Urysohn lemma).

Let Ω be a locally compact Hausdorff space. For any $K \subseteq U \subseteq \Omega$ where K is compact and U is open there is continuous $h : \Omega \to [0, 1]$ with compact $\operatorname{supp}(h) \subseteq U$ and attaining 1 on K.



Cor. If Ω is locally compact Hausdorff, then $\overline{C_c(\Omega)} = C_0(\Omega)$. So $C_0(\Omega)$ **is a completion of** $C_c(\Omega)$ in the supremum norm.

Proof: Let $x \in C_0(\Omega)$. For each n put

$$K_n := \{t : |x(t)| \geqslant 1/n\}, \;\; U_n := \{t : |x(t)| > 1/(n+1)\}.$$

Then K_n is a compact subset of an open set U_n. By **Urysohn lemma** there is $h_n \in C_c(\Omega)$ such that $0 \leqslant h \leqslant 1$, $h_n|_{K_n} = 1$ and $\mathrm{supp}(h_n) \subseteq U_n$. Putting $x_n := x \cdot h_n$ we get $x_n \in C_c(\Omega)$ and

$$\|x_n - x\|_\infty = \sup_{t\in\Omega} |x(t)h(t) - x(t)| = \sup_{t\in\Omega\setminus K_n} |x(t)h(t) - x(t)| \leqslant 2/n.$$

Hence $\{x_n\}_{n=1}^{\infty} \subseteq C_c(\Omega)$ converges to $x \in C_0(\Omega)$. ■

Ex. For a discrete space $\Omega = \mathbb{N}$ the linear space $C_c(\Omega)$ can be indentified with the **space of finite sequences**:

$$c_{00} = \{x = (x(1), x(2), ..., x(N), 0, 0, ...) : N \in \mathbb{N}, x(k) \in \mathbb{F}\}.$$

In particular, c_{00} is a dense subspace of c_0.